Suggested solution of HW2

Chapter 3 Q12: By Residue formula, for any $N \geq |u|$

$$\frac{1}{2\pi i}\oint_{|z|=N+1/2} \frac{\pi \cot \pi z}{(u+z)^2}\, dz = \sum_{|n|\leq N+1/2} Res\left(\frac{\pi\cot\pi z}{(u+z)^2}, n\right) + Res\left(\frac{\pi\cot\pi z}{(u+z)^2}, -u\right).$$

At $z = n$,

$$\frac{d}{dz}\sin\pi z\bigg|_n = \pi(-1)^n.$$

Thus,

$$Res\left(\frac{\pi\cot\pi z}{(u+z)^2}, n\right) = \frac{1}{(u+n)^2}.$$

Also, direct checking yield

$$Res\left(\frac{\pi\cot\pi z}{(u+z)^2}, -u\right) = -\frac{\pi^2}{\sin^2\pi u}.$$

On the other hand, on the circle $|z| = N + 1/2$, write $\pi z = x + iy$, fix a small $\delta > 0$. If $|x| > \delta$, we have

$$|\cot\pi z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y} \leq \frac{1+\sinh^2 y}{C + \sinh^2 y} < C_1.$$

If $|x| \leq \delta$,

$$|\cot\pi z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y} \leq \frac{1+\sinh^2 y}{\sinh^2 y} < 2.$$

Thus,

$$\frac{1}{2\pi i}\oint_{|z|=N+1/2} \frac{\pi\cot\pi z}{(u+z)^2}\, dz \to 0 \text{ as } N \to \infty$$

which yield the desired result.

Chapter 4 Q1: (a) if $\xi \in \mathbb{R}$,

$$A(\xi) - B(\xi) = e^{2\pi i t}\int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}dx = 0.$$

(b) We first claim that $A(z)$ is differentiable at $z_0 \in \mathbb{H}$. For $z = x + iy$, $y > 0$,

$$A(z) = \int_{-\infty}^{t} f(\xi)e^{-2\pi i(x+iy)(\xi-t)}d\xi = \int_0^{\infty} f(\xi)e^{2\pi i(x+iy)\xi}d\xi$$
$$= \int_0^{+\infty} f(\xi)e^{2\pi i x}e^{-2\pi y\xi}d\xi.$$

Since f is of moderate decrease, and $y > 0$, $A(z)$ define a holomorphic function on $\mathbb{H}$. And

$$|A(z)| \leq \int_0^{+\infty}|f(\xi)|e^{-2\pi y\xi}d\xi \leq \int_0^{+\infty}\frac{A}{1+\xi^2}e^{-2\pi y\xi}d\xi \leq \frac{A}{2\pi y} \to 0 \text{ as } y \to \infty.$$

Thus, $A(z)$ is a bounded holomorphic function. Similar for $B(z)$. By Morera's Theorem, $F(z)$ is a bounded entire function, and thus is constant. By above inequality, $F(z) \equiv 0$.

(c) Putting $z = 0$, we have for each $t \in \mathbb{R}$

$$\int_{-\infty}^{t} f(x)\, dx = 0.$$

By continuity, $f \equiv 0$.

Chapter 4 Q2: If f is analytic in S_a, obviously $f^{(n)}$ is analytic in S_b for any $0 \le b < a$. It remains to show that there exists $B_n > 0$ such that

$$|f^{(n)}(x+iy)| \le \frac{B_n}{1+x^2} \text{ for all } x \in \mathbb{R} \text{ and } |y| < b.$$

Let $\delta = a - b$, $w \in S_b$. By Cauchy formula,

$$\begin{aligned}
\left|\frac{f^{(n)}(w)}{n!}\right| &= \frac{1}{2\pi}\left|\oint_{B(w,\delta)} \frac{f(z)}{(z-w)^{n+1}}\, dz\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(w+\delta e^{i\theta})|}{\delta^n} d\theta \\
&\le \frac{1}{2\pi\delta^n}\int_0^{2\pi} \frac{A}{1+[Re(w+\delta e^{i\theta})]^2} d\theta
\end{aligned}$$

If $|Re(w)| > 2\delta$,

$$\left|\frac{f^{(n)}(w)}{n!}\right| \le \frac{A}{\delta^n}\cdot\frac{1}{1-\delta^2+[Re(w)]^2/2} \le \frac{A}{\delta^n}\frac{1}{1+(Re(w))^2/4} \le \frac{4A}{\delta^n}\frac{1}{1+(Re(w))^2}.$$

If $|Re(w)| \le 2\delta$,

$$\left|\frac{f^{(n)}(w)}{n!}\right| \le \frac{A}{\delta^n}\cdot\frac{1+4\delta^2}{1+(Re(w))^2}$$

Choose $B_n = \dfrac{An!(1+4\delta^2)}{\delta^n}$.

Chapter 4 Q3: If $\xi < 0$, let γ be the curve composed of the upper semi circle of radius R from R to $-R$ and the straight line from $-R$ to R on the real axis. For R sufficiently large, by Residue formula, one can obtain

$$\frac{1}{2\pi i}\oint_\gamma \frac{a}{a^2+z^2}e^{-2\pi iz\xi}\, dz = \frac{1}{2i}e^{2\pi\xi a}.$$

On the other hand,

$$\oint_\gamma f(z)\, dz = \int_{-R}^{R} \frac{a}{a^2+x^2}e^{-2\pi ix\xi}\, dx + \int_0^\pi \frac{a}{a^2+R^2e^{2i\theta}}e^{-2\pi iR\xi(\cos\theta+i\sin\theta)}\, d\theta$$

And

$$\left|\int_0^\pi \frac{a}{a^2+R^2e^{2i\theta}}e^{-2\pi iR\xi(\cos\theta+i\sin\theta)}\,d\theta\right| \leq \int_0^\pi \frac{a}{R^2-a^2}e^{2\pi R\xi\sin\theta}\,d\theta \to 0 \text{ as } R\to\infty.$$

Result follows when we take R tends to $+\infty$. If $\xi \geq 0$, instead we consider the curve composed of lower semi cirle from $-R$ to R and the straight line from R to $-R$. And then argue as same as before.
The inversion can be checked by direct integration.

$$\int_0^{+\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi = \frac{1}{2\pi(a-ix)} \quad \text{and} \quad \int_{-\infty}^0 e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi = \frac{1}{2\pi(a+ix)}$$

Summing up yield the result.

Chapter 4 Q6: Follows from applying Poisson summation formula to $f = \dfrac{a}{a^2+x^2}$ and the result in Q3.

$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \sum_{n=1}^{\infty} e^{-2\pi an} + \sum_{n=-\infty}^{0} e^{2\pi an} = 1+2\sum_{n=1}^{\infty} e^{-2\pi an}$$
$$= \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = \coth \pi a.$$

Chapter 4 Q7: (a) If $\xi \leq 0$, using contour integral as in the first part of Q3, as the pole is in the lower half plane, we get

$$0 = \oint_\gamma f(z)e^{-2\pi iz\xi}\,dz = \int_{-R}^R \frac{1}{(x+\tau)^k}e^{-2\pi ix\xi}\,dx + \int_0^\pi \frac{1}{(\tau+Re^{i\theta})^k}e^{-2\pi i\xi R(\cos\theta+i\sin\theta)}\,d\theta.$$

while

$$\int_0^\pi \left|\frac{1}{(\tau+Re^{i\theta})^k}e^{-2\pi i\xi R(\cos\theta+i\sin\theta)}\right|\,d\theta \leq \int_0^\pi \frac{1}{(R-|\tau|)^k}e^{2\pi i\xi\sin\theta}\,d\theta \to 0.$$

Thus, $\hat{f}(\xi) = 0$ if $\xi \leq 0$.

If $\xi > 0$, using the contour as in the latter part of Q3. we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{(z+\tau)^k}e^{-2\pi iz\xi}\,dz = Res_{-\tau}(f(z)e^{-2\pi iz\xi}).$$

At $z=-\tau$,

$$\left.\frac{d^{k-1}}{dz^{k-1}}e^{-2\pi iz\xi}\right|_{-\tau} = (-2\pi i\xi)^{k-1}e^{2\pi i\tau\xi}.$$

Thus,

$$Res_{-\tau}(f(z)e^{-2\pi iz\xi}) = \frac{(-2\pi i\xi)^{k-1}e^{2\pi i\tau\xi}}{(k-1)!}$$

in which

$$\oint_\gamma \frac{1}{(z+\tau)^k}e^{-2\pi iz\xi}\,dz = -\frac{(-2\pi i)^k\xi^{k-1}e^{2\pi i\tau\xi}}{(k-1)!}.$$

But

$$\oint_\gamma \frac{1}{(z+\tau)^k} e^{-2\pi i z\xi}\, dz = \int_R^{-R} \frac{1}{(x+\tau)^k} e^{-2\pi i x\xi}\, dx + O(\frac{1}{R^k}).$$

Takeing $R \to \infty$ yields

$$\hat{f}(\xi) = \frac{(-2\pi i)^k}{(k-1)!} \xi^{k-1} e^{2\pi i \xi \tau}.$$

Then we apply Poisson summation formula to get the desired equality.

(b) Putting $k = 2$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = -4\pi^2 \sum_{m=1}^{\infty} m e^{2\pi i m \tau}$$

Using the equality

$$\sum_{m=1}^{\infty} m z^m = \frac{z}{(1-z)^2} \quad \text{for } |z| < 1.$$

Since $Im(\tau) > 0$, we can substitute $z = e^{2\pi i \tau}$ into the above equation. Thus,

$$-4\pi^2 \sum_{m=1}^{\infty} m e^{2\pi i m \tau} = -4\pi^2 \frac{e^{2\pi i \tau}}{(1 - e^{2\pi i \tau})^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Yes. Since both $\dfrac{\pi^2}{\sin^2(\pi z)}$ and $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}$ define a meromorphic function on $\mathbb{C}$ with pole at integers, and equal in value on the upper half plane. By identity theorem, they are equal.

For sake of completeness, I prove the identity theorem for meromorphic function here. It suffices to prove that they are equal on any compact set $\Omega \subset \mathbb{C}$. If f and g are two meromorphic functions on Ω such that $f(z) = g(z)$ on a nonempty open region U. As poles are isolated, there exists holomorphic function f_1, f_2, g_1, g_2 on interior of Ω such that

$$f(z) = \frac{f_1}{f_2}, \quad g(z) = \frac{g_1}{g_2} \quad \text{on } int(\Omega).$$

Define $h = f_1 g_2 - f_2 g_1$ which is holomorphic on $int(\Omega)$. $h = 0$ on U. By identity theorem, $h = 0$ on Ω. Thus, $f \equiv g$ on Ω.

Chapter 5 Q2: (a) Let $p(z)$ be a polynomial of degree m. There exists a constant $C > 0$ such that

$$|p(z)| \le C(|z|^m + 1)$$

$\forall \epsilon > 0$, there exists $A_\epsilon > 0$ such that

$$|z|^m \le A_\epsilon exp(|z|^\epsilon)$$

Thus, $p(z)$ is of order less than or equal to ϵ, for any $\epsilon > 0$. So, it is of order 0.

(b) If $z = |z|e^{i\theta}$, $b = |b|e^{i\phi}$,

$$|e^{bz^n}| = e^{|z|^n|b|\cos(n\theta+\phi)} \le e^{|z|^n|b|}$$

So it is of order less than or equal to n. Put $z = x \in \mathbb{R}$ to see that the order is exactly n.

(c) Put $z = x \in \mathbb{R}$. Since for any $s \in \mathbb{R}$, there exists $R > 0$ s.t. $e^x > x^s$ for all $x > R$. The function e^{e^z} has infinity order of growth.

Chapter 5 Q3:

$$\begin{aligned}\left|\sum_{n\in\mathbb{N}} e^{i\pi n^2\tau} e^{2\pi inz}\right| &\le \sum_{n\in\mathbb{N}} e^{-\pi n\cdot\mathrm{Im}(n\tau+2z)} \\ &\le \sum_{n\in\mathbb{N}} e^{-\pi n^2\cdot Im(\tau)} \cdot e^{2\pi|n||z|} \\ &\le e^{2\pi|z|^2} \sum_{n\in\mathbb{N}} e^{-\pi n^2\cdot Im(\tau)/2} \quad \text{(using AM-GM inequality)} \\ &\le Ce^{2\pi|z|^2}\end{aligned}$$

for some constant C depends on τ only. Thus it is of order ≤ 2. It remains to show that the order is exactly 2. For each r and $m \in \mathbb{N}$, we have

$$\Theta(r + m\tau|\tau) = e^{-i\pi m^2 r - 2im\pi r}\Theta(r|\tau).$$

Thus,

$$|\Theta(r + m\tau|\tau)| = e^{t\pi m^2}|\Theta(r|\tau)|.$$

Choose r such that Right hand side is non-zero. Thus the growth of order is at least 2.

Chapter 5 Q6: Using the product formula for the sine function. Putting $z = 1/2$, we get

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}(1 - \frac{1}{4n^2}) = \prod_{n=1}^{\infty} \frac{(2n+1)(2n-1)}{(2n)^2}.$$

Chapter 5 Q7: (a) Clearly, $a_n \neq -1$ for all $n \in \mathbb{N}$, otherwise the conclusion fail. Since $|a_n| \to 0$, we may assume $|a_n| \le 1/2$ for all $n \in \mathbb{N}$. Hence, $\log(1 + a_n)$ is well defined if we choose the principle branch. By the power series expansion of log, we know that for all $|z| \le 1/2$,

$$|\log(1 + z) - z| \le C|z|^2.$$

Therefore, for any $m \ge n \ge 1$,

$$\left|\sum_{k=n}^{m} \log(1 + a_k) - \sum_{k=n}^{m} a_k\right| \le C \sum_{k=n}^{m} |a_k|^2$$

The conclusion follows immediately from cauchy criterion.

(b) Take $a_n = e^{in\pi/4}$ will suffices.

(c) Take a sequence a_n which contains a constant subsequence $a_{n_k} = -1$.

Chapter 5 Q8: The product converge since

$$\prod_{k=1}^{N} \cos \frac{z}{2^k} = \frac{\sin z}{2^N} \left[\sin \frac{z}{2^N}\right]^{-1} \to \frac{\sin z}{z} \quad \text{as } N \to \infty.$$

Chapter 5 Q9:

$$\prod_{k=0}^{N} (1 + z^{2^k})(1 - z) = (1 + z^{2^{N+1}}) \to 1 \quad \text{as } N \to \infty.$$

Extra question: Consider the contour suggested. Choose the log using Principle branch and use Residue theorem, we obtain

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{z^{a-1}}{1+z}\, dz = e^{i\pi(a-1)}.$$

On γ_1,

$$\int_{\gamma_1} \frac{z^{a-1}}{1+z}\, dz = \int_{\epsilon}^{R} \frac{e^{(a-1)\log(t+i\delta)}}{1+t+i\delta} dt \to \int_{\epsilon}^{R} \frac{t^{a-1}}{1+t}\, dt \quad \text{as } \delta \to 0.$$

Similarly,

$$\begin{aligned}
\int_{\gamma_3} \frac{z^{a-1}}{1+z}\, dz &= \int_{\epsilon}^{R} \frac{e^{(a-1)\log(t-i\delta)}}{1+t-i\delta} dt \\
&= \int_{\epsilon}^{R} \frac{e^{(a-1)\log\overline{(t+i\delta)}}}{\overline{1+t+i\delta}} dt \\
&= e^{(a-1)2\pi i} \int_{\epsilon}^{R} \frac{e^{(a-1)\overline{\log(t+i\delta)}}}{\overline{1+t+i\delta}} dt \to e^{(a-1)2i\pi} \int_{\epsilon}^{R} \frac{t^{a-1}}{1+t}\, dt \quad \text{as } \delta \to 0.
\end{aligned}$$

Therefore,

$$\int_{\gamma_1-\gamma_3} \frac{z^{a-1}}{1+z}\, dz = \left(\int_{\epsilon}^{R} \frac{t^{a-1}}{1+t}\, dt\right)(1 - e^{(a-1)2\pi})$$

while when $R \to \infty$ and $\epsilon \to 0$,

$$\left|\int_{\gamma_2} \frac{z^{a-1}}{1+z}\, dz\right| \le CR \cdot R^{a-1} \cdot \frac{1}{1+R} \to 0$$

and

$$\left|\int_{\gamma_2} \frac{z^{a-1}}{1+z}\, dz\right| \le C\epsilon \cdot \epsilon^{a-1} \cdot \frac{1}{1-\epsilon} \to 0.$$

Hence,

$$\int_0^{\infty} \frac{t^{a-1}}{1+t}\, dt = \frac{2\pi i e^{i\pi(a-1)}}{(1 - e^{(a-1)2\pi i})} = \frac{\pi}{\sin(\pi a)}.$$